Some explicit compact generators for $D_{lis}(\mathrm{Bun}_G, \Lambda)$

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Workshop on p-adic Hodge theory

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$* \; G/\mathbb{Q}_p$ reductive

$\Lambda = \mathbb{Z}_\ell$-algebra, $\ell \neq p$ (coefficients)

$D(G(\mathbb{Q}_p), \Lambda)$ smooth rep. of $G(\mathbb{Q}_p)$ with values in Λ-modules

derived

Compact generators $\left(c\text{-}\mathrm{Ind}_K^{G(\mathbb{Q}_p)} \Lambda\right)_{K \subset G(\mathbb{Q}_p) \text{ compact open pro-}p}$

$\rightarrow \mathrm{Hom}\left(c\text{-}\mathrm{Ind}_K^{G(\mathbb{Q}_p)} \Lambda, \pi\right) = \pi^K$

$*$ Geometric interpretation

$* = \mathrm{Spa}\, \overline{\mathbb{F}}_p$ final object of v-topos on $\mathrm{Perf}_{\overline{\mathbb{F}}_p}$

$\boxed{\Lambda \text{ torsion from now to simplify}}$

$$f_K : [*/\underline{K}] \longrightarrow [*/\underline{G(\mathbb{Q}_\ell)}]$$

$$D(G(\mathbb{Q}_\ell), \Lambda) = D_{ét}(\underbrace{[*/\underline{G(\mathbb{Q}_\ell)}]}, \Lambda)$$

Artin v-stack coho. smooth dim. 0

$$\text{c-Ind}_K^{G(\mathbb{Q}_\ell)} \Lambda \longleftrightarrow f_{K!} \Lambda$$

$*$ $\quad \mathrm{Bun}_G$ = Artin v-stack of G-bundles/curve

$$\begin{array}{c} \mathrm{Bun}_G \\ \downarrow \\ * \end{array}$$

Recall $D_{ét}(\mathrm{Bun}_G, \Lambda)$ has a semi-orthogonal decomposition

by $\left(D(G_b(\mathbb{Q}_\ell), \Lambda) \right)_{[b] \in \underbrace{B(G)}}$

Kottwitz set $G(\breve{\mathbb{Q}}_\ell)/b \sim g b \sigma(g)^{-1}$

Ex: $b = 1$. $G_b = G$

$$\begin{tikzcd} {[*/G(\mathbb{Q}_p)]} \arrow[r, hook, "i^1", "\text{open}"'] \arrow[d, phantom, "\parallel"] & \mathrm{Bun}_G \\ \mathrm{Bun}_G^{ss,\, c_1=0} & \end{tikzcd}$$

$$D(G(\mathbb{Q}_p), \Lambda) = D_{\acute{e}t}([*/G(\mathbb{Q}_p)], \Lambda) \underset{(i^1)^*}{\overset{(i^1)_!}{\rightleftarrows}} D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$$

Idea: Good objects for the local Langlands program are not in $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$

$\to$ Construct compact generators of $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$ that generalize the preceding one of $D(G(\mathbb{Q}_p), \Lambda)$.

Jacobian criterion of smoothness

$S \in \mathrm{Perf}_{\bar{\mathbb{F}}_p}$

$$\begin{array}{l} Z \\ \downarrow \ \text{smooth map of sous-perfectoid spaces} \\ X_S \end{array}$$

$\rightsquigarrow$ relative curve associated to S

$$\exists \quad \begin{tikzcd} Z \arrow[r, hook, "\text{open in Zariski closed}"] \arrow[d] & \mathbb{P}^n_{X_S} \arrow[dl] \\ X_S & \end{tikzcd}$$

quasi-projectivity hypothesis

Def: $\mathcal{M}_Z$ = moduli of sections of Z/X_S

$$T/S \longmapsto \left\{ \text{sections } s \quad \begin{tikzcd} & Z \arrow[d] \\ X_T \arrow[ur, "s"] \arrow[r] & X_S \end{tikzcd} \right\}$$

Fact: $\mathcal{M}_Z$ locally spatial diamond

$$\begin{tikzcd} \mathcal{M}_Z \arrow[d] \\ S \end{tikzcd}$$

algebraic equations in BC spaces

Ex (linear case). $\mathcal{E}$ v.b. $/X_S$, $Z = \mathbb{V}(\mathcal{E})$

$$\mathcal{M}_Z = BC(\mathcal{E})$$

Def: $\mathcal{M}_Z^{sm} \underset{\text{open}}{\subset} \mathcal{M}_Z$

$\quad\;\; \|$

$\left\{ \text{sections } s \text{ s.t. } \underbrace{s^* T_{Z/X_S}}_{\text{v.b. on } X_T} \text{ has} > 0 \text{ H.N. slopes} \right\}$

[**Th**: $\mathcal{M}_2^{sm} \to S$ is ℓ-coho. smooth.] Very deep result

Ex (linear case): $\mathcal{E}/X_S$ v.b., $U \subset S$ open where $\mathcal{E}$ has >0 H.N. slopes. Then $BC(\mathcal{E})_{|U} \to U$ is ℓ-coho. smooth.

Ex. $\mathcal{E}$ = étale G-torsor on X_S

$Z = P\backslash\mathcal{E} \to X_S$.

$\mathcal{M}_2$ = moduli of reductions of $\mathcal{E}$ to P

$\mathcal{M}_2^{sm} = \{$reductions $\mathcal{E}_P$ s.t. $\mathcal{E}_P \overset{P}{\times} \mathfrak{g}/\mathfrak{p}$ has >0 H.N. slopes$\}$

$\downarrow$ ℓ-coho. smooth

S

$\Rightarrow$ [$\underbrace{Bun_P^{>0}} \to Bun_G$ is ℓ-coho. smooth]

P-torsors $\mathcal{E}_P$ s.t. $\mathcal{E}_P \overset{P}{\times} \mathfrak{g}/\mathfrak{p}$ has >0 H.N. slopes

Ex. $G = GL_n$ $\quad$ P = Borel = B

$Bun_B^{>0}$ = v.b. $\mathcal{E}$ of rk. n + filtration $0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_n = \mathcal{E}$ full flag

$\deg(\mathcal{E}_1/\mathcal{E}_0) < \deg(\mathcal{E}_2/\mathcal{E}_1) < \dots < \deg(\mathcal{E}_n/\mathcal{E}_{n-1})$

"anti-HN filtration"

The moduli $\mathcal{M}_b$

$$[b] \in B(G) = |\mathrm{Bun}_G|$$

G quasi-split to simplify

M_b = Centralizer of ν_b slope morphism $\in X_*(A)^+_{\mathbb{Q}}$

$\cap$

P_b standard parabolic

Can suppose $b = b_{M_b} \in M_b(\breve{\mathbb{Q}}_p)$ basic

Def: $\mathcal{M}_b$ = moduli of P_b-bundles $\mathcal{E}$ s.t. $\mathcal{E} \overset{P_b}{\times} M_b$ is geo. fiberwise isomorphic to $\mathcal{E}_{b_{M_b}}$ as a M_b-bundle

Ex. $G = GL_n$, $d_1 < \dots < d_n$ $\quad b = \begin{pmatrix} p^{-d_1} & & \\ & \ddots & \\ & & p^{-d_n} \end{pmatrix}$

$\mathcal{M}_b$ = moduli of v. b. $\mathcal{E}$ + full flag $0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_n = \mathcal{E}$
s.t. $\deg(\mathcal{E}_i / \mathcal{E}_{i-1}) = d_i$.

$$\begin{tikzcd}
 & {[\underline{G_b(\mathbb{Q}_p)} \backslash \widetilde{\mathcal{F}}_b]} \arrow[d, phantom, "\shortparallel"] & \\
\widetilde{\mathcal{M}}_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}_b \arrow[r, "\pi_b"] \arrow[d, "q_b"] & \mathrm{Bun}_G \\
\ast \arrow[r] \arrow[u, bend left] & {[\ast / \underline{G_b(\mathbb{Q}_p)}]} \arrow[u, bend right, "\text{given by } \mathcal{E}_b"'] &
\end{tikzcd}$$

$$q_b : \mathcal{E} \mapsto \mathcal{E} \overset{P_b}{\times} \mathcal{M}_b$$

$$\pi_b : \mathcal{E} \mapsto \mathcal{E} \overset{P_b}{\times} G$$

* Jacobian criterion $\Rightarrow$ π_b ℓ-coho. smooth

* $\underline{\mathrm{Th}}$ $\widetilde{\mathcal{F}}_b \setminus \{\ast\}$ is a $\overset{\text{q.c.}}{\overbrace{\text{spatial}}}$ diamond.

  $\rightarrow$ Artin criterion for spatial diamonds.

$\Rightarrow$ $\underline{\underline{\mathrm{Th}}}$: $\forall A \in D_{\text{ét}}(\widetilde{\mathcal{F}}_b, \Lambda)$, $R\Gamma(\widetilde{\mathcal{F}}_b, A) = i^* A$

where $i : \ast \hookrightarrow \widetilde{\mathcal{F}}_b$

$K \subset G_b(\mathbb{Q}_p)$ compact open pro-p

$$\begin{tikzcd}
\left[\underline{K}\backslash \widetilde{\mathcal{M}}_b\right] \arrow[rr, "f_K^b"] \arrow[dr, "\text{étale}"'] & & \mathrm{Bun}_G \\
& \mathcal{M}_b \arrow[ur, "\pi_b"'] &
\end{tikzcd}$$

f_K^b: ℓ-cohomologically smooth

π_b: ℓ-cohomologically smooth

Def: $A_K^b = f^b_{K!} f_K^{b!} \Lambda \in D_{et}(\mathrm{Bun}_G, \Lambda)$

$\left[\text{Th: } \forall B \in D_{et}(\mathrm{Bun}_G, \Lambda),\ R\mathcal{H}om(A_K^b, B) = \left[\underbrace{(i^b)^* B}_{\in D(G_b(\mathbb{Q}_p), \Lambda)}\right]^K \in D(\Lambda) \right]$

$i^b : [*/\widetilde{G}_b] \hookrightarrow \mathrm{Bun}_G$

$\Rightarrow \left[(A_K^b)_{[b], K \subset G_b(\mathbb{Q}_p)} \text{ set of compact generators} \right]$