

Some explicit compact generators

for $D_{\text{lis}}(\text{Ban}_G, \Lambda)$

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Workshop on p -adic
Hodge theory

①

* G/\mathbb{Q}_p Reductive

$\Lambda = \mathbb{Z}[\ell]$ -algebra, $\ell \neq p$ (coefficients)

$D(G(\mathbb{Q}_p), \Lambda)$
 \uparrow
derived

smooth rep. of $G(\mathbb{Q}_p)$ with
values in Λ -modules

Compact generators $(\text{c-Ind}_K^{G(\mathbb{Q}_p)} \Lambda)_{K \subset G(\mathbb{Q}_p)}$ compact
open pro- p

$$\rightarrow \text{Hom}(\text{c-Ind}_K^{G(\mathbb{Q}_p)} \Lambda, \pi) = \pi^K$$

* Geometric interpretation

* = $\text{Spa } \overline{\mathbb{F}_p}$ final object of
 v -topos on $\text{Perf } \overline{\mathbb{F}_p}$

Λ torsion from
now to simplify

$$f_k: [*/\underline{k}] \longrightarrow [*/\underline{G(\mathbb{Q}_f)}]$$

$$D(G(\mathbb{Q}_f), \Lambda) = D_{\text{ét}}([*/\underline{G(\mathbb{Q}_f)}], \Lambda)$$

Artin v. stacks Coh. smooth
dim. 0

$$C\text{-Ind}_k^{G(\mathbb{Q}_f)} \Lambda \longleftrightarrow f_{k!} \Lambda$$

* $\text{Bun}_G = \text{Artin v. stacks of } G\text{-bundles/curve}$
 \downarrow
 *

Recall $D_{\text{ét}}(\text{Bun}_G, \Lambda)$ has a semi-orthogonal decomposition

by $\left(D(G(\mathbb{Q}_f), \Lambda) \right)_{[g] \in B(G)}$

thought set $G(\mathbb{Q}_f)/\text{brng } g^{-1}$

Ex: $b=1$. $G_x = G$

(2)

$$[* / G(\mathbb{Q}_p)] \xrightarrow[\text{open}]{i_1} \text{Bun}_G$$

\parallel
 $\mathcal{H}, c_1=0$
 Bun_G

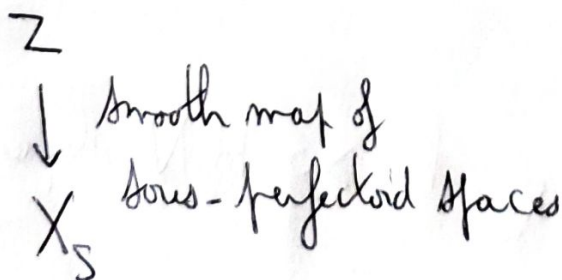
$$D(G(\mathbb{Q}_p), 1) = D_{\text{ét}}([* / G(\mathbb{Q}_p)], 1) \xrightleftharpoons[(i_1)^*]{(i_1)!} D_{\text{ét}}(\text{Bun}_G, 1)$$

Idea: Good objects for the local Langlands program are not in $D_{\text{ét}}(\text{Bun}_G, 1)$

→ Construct compact generators of $D_{\text{ét}}(\text{Bun}_G, 1)$ that generalize the preceding one of $D(G(\mathbb{Q}_p), 1)$.

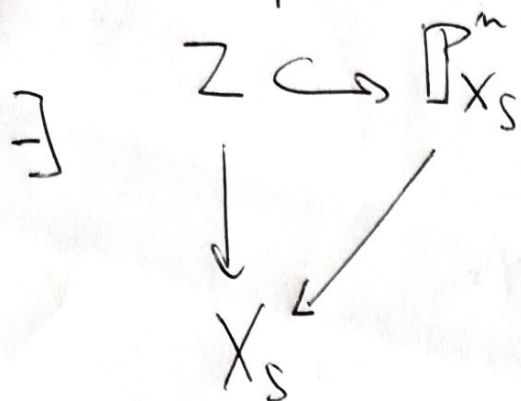
Jacobian criterion of smoothness

~~$S \in \text{Perf}_{\mathbb{F}_p}$~~ $S \in \text{Perf}_{\mathbb{F}_p}$



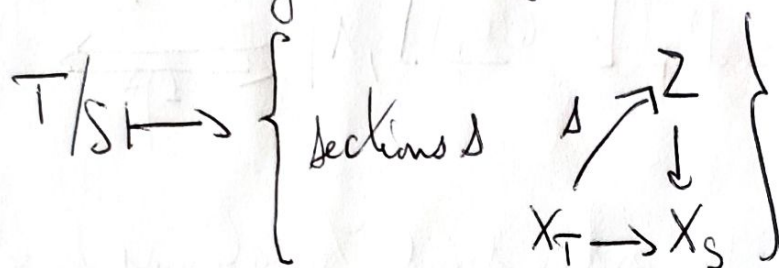
$\underbrace{\quad}_{\text{relative curve associated to } S}$

open in Zariski closed



quasi-projectivity hypothesis

Def: $\mathcal{M}_2 =$ moduli of sections of Z/X_S



Fact: \mathcal{M}_2 locally spatial diamond

algebraic equations
in BC spaces



Ex (linear case). \mathcal{E} v.l. / X_S , $Z = V(\mathcal{E})$

$$\mathcal{M}_2 = BC(\mathcal{E})$$

Def: $\mathcal{M}_2^{\text{sm}} \subseteq \mathcal{M}_2$
open

||
{ sections s s.t. $s^* T_{Z/X_S}$ has > 0 H.N. slope }

$\underbrace{\hspace{2cm}}$
v.l. on X_T

$\left[\text{Th: } \mathcal{M}_2^{\text{sm}} \rightarrow S \text{ is l-cho. smooth.} \right]$ Very deep result

\mathcal{E}_n (linear case): \mathcal{E}/X_S v.b., $\cup_{\text{open}} S'$ where \mathcal{E} has >0 H.N. slopes. Then $BC(\mathcal{E})|_U \rightarrow U$ is l-cho. smooth.

\mathcal{E}_n : $\mathcal{E} = \text{stable } G\text{-torsor on } X_S$

$Z = \mathbb{P}^{\mathcal{E}} \rightarrow X_S$

$\mathcal{M}_2 = \text{moduli of reductions of } \mathcal{E} \text{ to } P$

$\mathcal{M}_2^{\text{sm}} = \{ \text{reductions } \mathcal{E}_P \text{ s.t. } \mathcal{E}_P \times \mathbb{G}/P \text{ has } >0 \text{ H.N. slopes} \}$

\downarrow l-cho. smooth
 S

$\Rightarrow \left[B_{\text{sm}}^P \rightarrow B_{\text{sm}}^G \text{ is l-cho. smooth} \right]$

$\underbrace{\hspace{10em}}_{P\text{-torsors } \mathcal{E}_P \text{ s.t. } \mathcal{E}_P \times \mathbb{G}/P \text{ has } >0 \text{ H.N. slopes}}$

\mathcal{E}_n : $G = GL_n$ $P = \text{Borel} = B$ \nearrow full flag

$B_{\text{sm}}^G = \text{v.b. } \mathcal{E} \text{ of rank } n + \text{filtration } 0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_n = \mathcal{E}$
 $\deg(\mathcal{E}_1/\mathcal{E}_0) < \deg(\mathcal{E}_2/\mathcal{E}_1) < \dots < \deg(\mathcal{E}_n/\mathcal{E}_{n-1})$

"anti-HN filtration"

The moduli \mathcal{M}_b

$$[b] \in B(G) = |Bun_G|$$

G quasi-split & simply

$M_b =$ Centralizer of γ_b slope morphism $\in X_*(A)_{\mathbb{Q}}^+$

$\bigcap P_b$ standard parabolic

Can suppose $b \in M_b(\mathbb{Q})$ basic

Def: $\mathcal{M}_b =$ moduli of P_b -bundles \mathcal{E} s.t. $\mathcal{E} \times_{M_b}^{P_b}$

is geo. fiberwise isomorphic to $\mathcal{E}_{\text{triv}}$ as a M_b -bundle

Ex. $G = GL_n$, $d_1 < \dots < d_n$

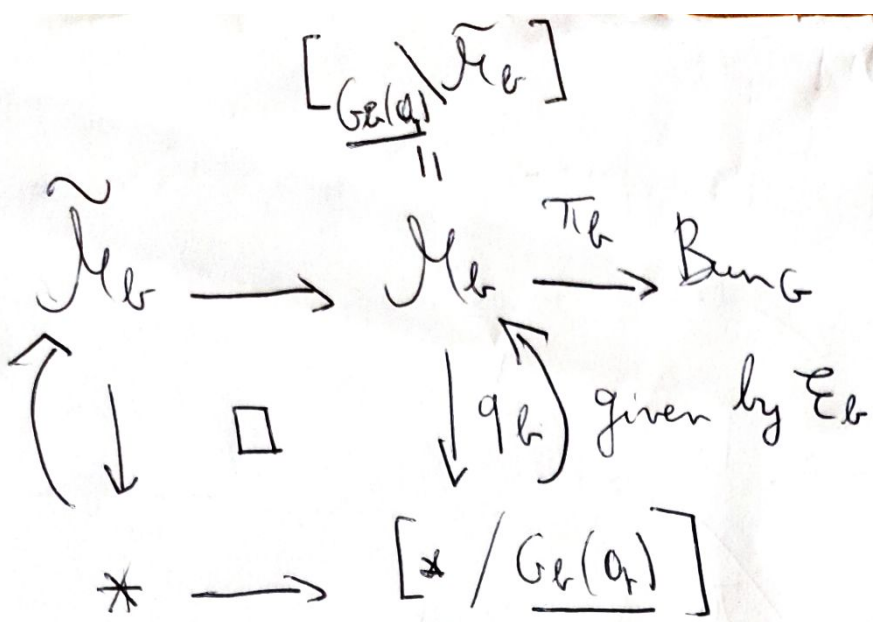
$$b = \begin{pmatrix} \uparrow^{-d_1} & & \\ & \ddots & \\ & & \uparrow^{-d_n} \end{pmatrix}$$

$\mathcal{M}_b =$ moduli of v.b. \mathcal{E} + full flag

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \dots \subsetneq \mathcal{E}_n = \mathcal{E}$$

s.t. $\deg(\mathcal{E}_i/\mathcal{E}_{i-1}) = d_i$.

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$$q_b: \mathcal{E}_1 \rightarrow \mathcal{E} \times^{P_b} M_b$$

$$\pi_b: \mathcal{E} \rightarrow \mathcal{E} \times^{P_b} G$$

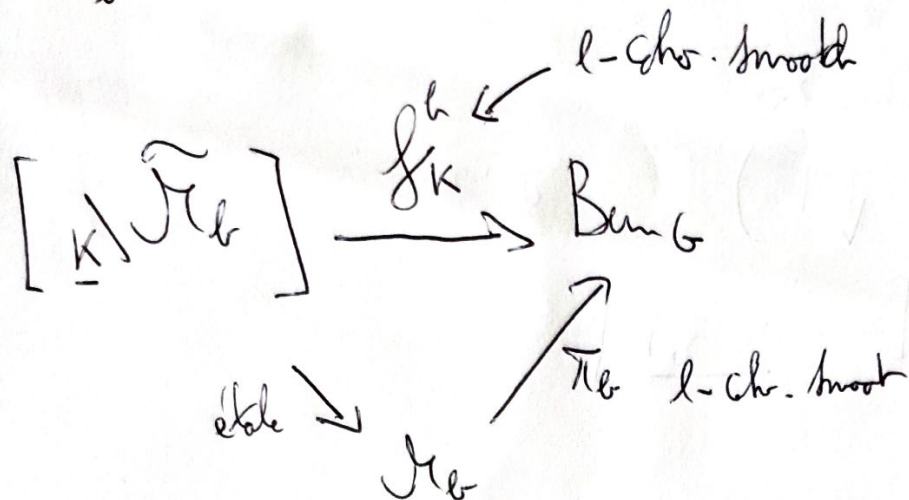
* Jacobian criterion $\Rightarrow \pi_b$ l-coho. smooth

* Th. $\tilde{M}_b \setminus \{*\}$ is a spatial diamond.

\rightarrow Artin criterion for spatial diamonds.

\Rightarrow Th. $\forall A \in \text{Det}(\tilde{F}_b, \lambda)$, $\text{RP}(\tilde{F}_b, A) = i^* A$
 where $i: \tilde{M}_b \hookrightarrow \tilde{F}_b$

$K \subset G(\mathbb{Q})$ Compact open pro-f



Def: $A_K^h = f_K^h \circ f_K^h! \wedge \in \text{Det}(\text{Bun}_G, A)$

Th: $\forall B \in \text{Det}(\text{Bun}_G, \wedge), \text{RHom}(A_K^h, B) = \underbrace{[(i^h)^* B]^k}_{\in \text{Det}(G_G(\mathbb{Q}), \wedge)} \in \text{D}(A)$

$i^h: [* / G_G] \hookrightarrow \text{Bun}_G$

$\Rightarrow \left[(A_K^h)_{[K], K \subset G_G(\mathbb{Q})} \text{ set of compact generators} \right]$